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On the structure of inertial waves produced by an obstacle in a deep, rotating container

By K. STEWARTSON
Department of Mathematics, University College London
AND H. K. CHENG
Department of Aerospace Engineering, University of Southern California, Los Angeles

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The inviscid flow above an obstacle in slow transverse motion inside a rotating vessel is analysed to study the influence of the container depth on the basic steady flow structure. An asymptotic theory is presented for an arbitrarily small Rossby number $Ro = U_0/2\Omega l$ under a fixed $H = hRo/l$ (where Ω is the angular velocity of the container, U_0 the obstacle velocity relative to the vessel, h the depth of the container, and l a body length measured transversely to the rotation axis). The equations when linearized for a thin obstacle or shallow topography take on the form of the inertial-wave equation; their solutions for non-vanishing H are obtained for obstacles of three-dimensional as well as ridge-like two-dimensional shapes. In all cases analysed, the solution possesses a *bimodal* structure, of which one part is column-like with a vorticity proportional to the body elevation (or ground topography). The other part is confined mainly to a region enclosing the body, extending a distance $O(H^{\frac{1}{2}})$ upstream of the obstacle and behind a wedge-shaped caustic front at large distances; its contribution consists of lee waves similar to that discussed by Cheng (1977) for an infinite depth. The field associated with the lee waves is then biased on the downstream side, but there is little indication of any tendency to tilting in the sense of Hide, Ibbetson & Lighthill (1968).

1. Introduction

The slow steady transverse motion of an inviscid fluid past a body in a rotating system is usually thought of as satisfying the Taylor–Proudman condition that the pressure is invariant along lines parallel to the axis of rotation and consequently to take on a columnar structure (Greenspan 1969). There are, however, a number of experiments in which the depth of the container plays a significant role (Hide & Ibbetson 1966; Hide, Ibbetson & Lighthill 1968, hereafter referred to as HIL; Mason 1975; Maxworthy 1977). A parameter, essentially the product of the ratio of the container depth to a dimension of the obstacle and the Rossby number Ro, which we shall refer to as H, has been identified as having a controlling influence on the flow structure. Further, linear unsteady theoretical studies which assume that the depth of the container is infinite (Grace 1926; Stewartson 1953) arrive at a different and incompatible conclusion about the steady state from that (Stewartson 1967) which takes into account the presence of an upper lid. The time scale of these theories is

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Ω^{-1}, where Ω is the angular velocity of the fluid, and it turns out that for deep containers a larger time scale is needed to achieve steady motion.

In this paper we shall derive a set of equations including nonlinear advective terms to describe the final approach to the steady state at arbitrary values of H whose solution might be expected to apply to containers of all depths. It then becomes apparent that the solution obtained earlier for infinite depth only refers to an intermediate time in the evolutionary process and the final steady state for H finite is much more complicated. These equations are also consistent with the quasi-geostrophic equations obtained by Ingersoll (1969) when $H \ll 1$. In order to obtain particular solutions we restrict attention to these obstacles or shallow topography when the equations may be linearized and reduce to a form already studied by Lighthill (1967), HIL, and more recently by Cheng (1977). The first two of these studies examined the propagation of waves in an infinite medium, and the last the disturbance produced by motion of a thin obstacle in a tank of infinite depth, the upper boundary being replaced by a radiation condition.

These studies are also of interest in connexion with an experimental result obtained in HIL that the Taylor column appears to be tilted, the angle of tilt being $\approx \frac{3}{2}Ro$, where Ro is the Rossby number of the flow. Using his general theory of wave propagation, Lighthill obtained the same estimate for the tilt, but, notwithstanding the remarkable agreement with experiment, some questions remain. In particular, the numerical factor was obtained using the solution of a different but related problem (Jacobs 1964; Stewartson 1967) for the far field in an infinite domain. Further, as Lighthill showed, the waves in the far field generated by the obstacle are confined within a wedge extending downstream (i.e. are lee waves) and Cheng's more detailed studies of their properties has revealed that their amplitude does not tend to zero with increasing distance downstream. This feature is compatible with the wave propagation analysis, but was not taken into account in the earlier studies of tilting.

Here we complete Cheng's arguments by adding an upper lid to the domain of flow and find that a two-dimensional topography produces a lee wave system which fills the entire region downstream and gives no sign of any tilting of preferred directions. For three-dimensional obstacles, we identify a *bimodal* structure to the steady-state solution. Of the two characteristics, one is columnar and partially resembles a Taylor column, and the other consists of a lee wave system surrounding the obstacles and extending downstream behind a wedge. The column of the first mode does not tilt; there is clearly a bias in the lee wave system in favour of the downstream side of the body, but it cannot be interpreted as a tilt in the sense used by HIL. Thus it appears that a more general and nonlinear theory is required to explain this phenomenon. The lee-wave system has been observed at moderate values of H in recent experiments by Maxworthy on the effect of ground topography and has qualitatively the structure described here.

2. Governing equations: slow uniform motion of obstacle in a deep rotating container

The weakly nonlinear equations

Consider the slow motion of a submerged, impermeable body at a uniform velocity U_0 relative to an inviscid fluid which is confined between two parallel planes, and rotating with angular velocity Ω about a perpendicular axis. We choose a Cartesian co-ordinate system (x^*, y^*, z^*) fixed to the body, with the two parallel planes located at $z^* = 0$ and $z^* = h$. Let the velocity components of the fluid be u, v, and w, corresponding to x^*, y^* and z^*. Only uniform body motions in the lower plane $z^* = 0$ are studied in this paper. The undisturbed fluid velocity at locations far from the obstacle may thus be taken as $(U_0, 0, 0)$. It is assumed that the transverse dimension of the obstacle, l, is small compared to the depth h, as is the relative speed $|U_0|$ compared to Ωl.

It is reasonable to stipulate that all velocity components are comparable to U_0, and the length scale characterizing variations in the transverse plane is comparable to l. The z^* scale characterizing the axial variation is taken to be h, inasmuch as the depth influence is the objective of our analysis. The perturbation pressure from the equilibrium value, $p^* - p_e^*$, is scaled by $2\rho U_0 \Omega l$ as in standard treatises on low Rossby number flows. Having in mind a formulation capable of establishing a steady-state solution in the limit $t^* \rightarrow \infty$, we must choose a time scale comparable to the transit time for the steady flow around the obstacle, l/U_0 (instead of Ω^{-1}). Thus, we introduce the following set of reduced variables for the present analysis:

$$\left.\begin{array}{l} x = x^*/l, \quad y = y^*/l, \quad z = z^*/h, \quad t = t^* U_0/l; \\ u = u^*/U_0, \quad v = v^*/U_0, \quad w = w^*/U_0; \\ p = (p^* - p_e^*)/2\rho U_0 \Omega l. \end{array}\right\} \tag{2.1}$$

Then the governing equations reduce to

$$-v + Ro \frac{D}{Dt} u = -\frac{\partial p}{\partial x}, \tag{2.2a}$$

$$u + Ro \frac{D}{Dt} v = -\frac{\partial p}{\partial y}, \tag{2.2b}$$

$$Ro \frac{D}{Dt} w = -\frac{l}{h} \frac{\partial p}{\partial z}, \tag{2.2c}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = -\frac{l}{h} \frac{\partial w}{\partial z}, \tag{2.2d}$$

where
$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} + w \frac{\partial}{\partial z},$$

and $Ro \equiv U_0/2\Omega l$ is the Rossby number, assumed to be much smaller than unity. The impermeable surface of the obstacle is prescribed as $z^* = \epsilon l f(x, y)$, where ϵl characterizes the height of the obstacle and is at present an arbitrary number. The appropriate boundary conditions are [with $r \equiv (x^2 + y^2)^{\frac{1}{2}}$]

$$w = \epsilon \left(u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} \right) f, \quad \text{at} \quad z = \epsilon \frac{l}{h} f; \tag{2.3a}$$

$$w = 0 \quad \text{at} \quad z = 1; \tag{2.3b}$$

$$(u, v, w) \rightarrow (1, 0, 0) \quad \text{as} \quad r \rightarrow \infty, \quad t \neq \infty. \tag{2.3c}$$

An initial condition at $t = 0$, say, must be prescribed. We note in passing that (2.3c) precludes neither lee waves with undiminishing strength nor an upstream influence, if any, in the steady-state limit $t \to \infty$.

Were we to omit the inertial terms associated with the Rossby number from (2.2), we would have

$$v = \frac{\partial p}{\partial x}, \quad u = -\frac{\partial p}{\partial y}, \quad \frac{\partial w}{\partial z} = 0, \quad \frac{\partial p}{\partial z} = 0. \tag{2.4}$$

These results are, of course, formally equivalent to the Proudman–Taylor theorem, and we notice at once that a contradiction with the two boundary conditions for w exists, except if $\epsilon = 0$, or if the obstacle has a compact support† over which a pure Taylor column with trapped fluid may stand. This possible contradiction and the lack of uniqueness [in the vorticity as implied by (2.4)] can be resolved if one includes the inertial terms (the unsteady and advective terms $\partial/\partial t + u\,\partial/\partial x$, in particular). The latter inclusion is permissible for a deep container, for which l/h is comparable to Ro [cf. (2.2c)]. We may accordingly analyse the problem as an asymptotic theory for the single limit $Ro \to 0$ under a fixed parameter

$$H \equiv Ro\frac{h}{l} = \frac{U_0 h}{2\Omega l^2}. \tag{2.5}$$

A parameter of this type was first realized by Hide & Ibbetson (1966) in experimental studies of Taylor columns, namely ϵ/H, defined as $\mathscr{S}$ in Mason's (1975) work.

Assuming that all reduced variables are of unit order as $Ro \to 0$, with H fixed, we recover from (2.2) the familiar linear geostrophic relations among u, v, and p to the leading order

$$u = -\partial p/\partial y, \tag{2.6a}$$

$$v = \partial p/\partial x. \tag{2.6b)‡}$$

Using these to eliminate u and v, we obtain from (2.2c)

$$\partial p/\partial z = -H\left[\frac{\partial w}{\partial t} + \frac{\partial(p, w)}{\partial(x, y)}\right], \tag{2.7a}$$

and from (2.2d)

$$\frac{\partial w}{\partial z} = H\left[\frac{\partial}{\partial t}(\nabla_1^2 p) + \frac{\partial(p, \nabla_1^2 p)}{\partial(x, y)}\right]. \tag{2.7b}$$

where $\nabla_1^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$. The right-hand members of the last two equations are generally nonlinear, signifying a departure from the Proudman–Taylor theorem (2.4), and their contribution is now seen to be controlled by the parameter H. The remainders in (2.7) may be seen to be of order HRo, and the terms omitted from (2.6) are $O(Ro)$. Therefore, the equations are valid for *all depths*, so long as the reduced variables and their derivatives remain of unit order. The boundary condition for w on the obstacle in (2.3a) can now be transformed to the plane $z = 0$ as

$$w = \epsilon\frac{\partial(p, f)}{\partial(x, y)} \quad \text{at} \quad z = 0 \tag{2.8a}$$

† That is, the elevation must vanish beyond a certain distance from the obstacle.

‡ The terms omitted on the right of (2.6a) and (2.6b) are, respectively,

$$-Ro[p_{xt} + \partial(p, p_x)/\partial(x, y)] + O(Ro^2, Ro^2/H)$$

and

$$-Ro[p_{yt} + \partial(p, p_y)/\partial(x, y)] + O(Ro^2, Ro^2/H).$$

with an error comparable to $z\,\partial p/\partial z$ at the surface, i.e. $O(\epsilon Ro)$. The other conditions of (2.3) are unaffected and repeated here for convenience:

$$w = 0 \quad \text{at} \quad z = 1, \tag{2.8b}$$

$$(p_x, p_y, w) \rightarrow (0, -1, 0) \quad \text{as} \quad r \rightarrow \infty, \quad t < \infty. \tag{2.8c}$$

With prescribed initial data, (2.7) along with boundary condition (2.8) can be solved for p and w.

It is clear from (2.5)–(2.8) that a formal degeneracy resulting from an unbounded H can be eliminated by taking a new axial variable $\hat{z} = Hz$; except for a slightly different definition of the Rossby number (Cheng used $\mathscr{R} = U_0/\Omega l$), and writing Θ for H, (2.5)–(2.8) in body-fixed co-ordinates and those given in Cheng (1977) in container-fixed co-ordinates are completely equivalent.

As $H \rightarrow 0$, it appears, and will be confirmed by more detailed analysis below when $\epsilon/H \ll 1$, that the columnar structure fills the entire depth of the container. However, $\partial w/\partial z$ is not necessarily zero as in (2.4). In the steady state

$$w = \epsilon(1-z)\frac{\partial(f,p)}{\partial(x,y)}, \quad \nabla_1^2 p = -\frac{\epsilon f}{H} + g(p) \tag{2.9}$$

is a consistent solution of (2.7) and (2.8) where g is an arbitrary function. Thus our basic theory reduces to Ingersoll's (1969) quasi-geostrophic model in the limit $H \rightarrow 0$. If all streamlines originate in a region of uniform flow $g = 0$ and then it may be shown (Huppert 1975) that this assumption is self-consistent provided $\epsilon \leqslant O(H)$. For larger values of ϵ closed streamlines appear above the obstacle and g is then partially indeterminate. At finite values of ϵ and for a finite obstacle g may be found either by an appeal to an initial value problem (Stewartson 1967) or viscosity (Jacobs 1964); in each case the trapped fluid is stagnant and

$$w \equiv 0. \tag{2.10}$$

These flow properties are broadly observed in the experiments of Hide & Ibbetson (1966) and Maxworthy (1970).

Such a structure of the steady state solution of (2.7) and (2.8) is also possible when $H \neq 0$ for finite obstacles, but is unlikely to occur in practice. For as we shall show below, a lee wave structure is set up in this case even when $\epsilon \ll 1$ generating motions violating (2.10).

Another form of columnar structure was obtained for transverse flow of a sphere in an unbounded fluid (Grace 1926; Stewartson 1953). When $\Omega t^* \gg 1$ the relative motion appears to be generally steady with a continuous transverse velocity distribution and mass flux through the column. However, the governing equations they used do not include the advective terms of (2.7) and it was tacitly assumed that $U_0 t^* \ll l$. Thus it is only relevant to an intermediate stage in the evolution of the flow field. The idea underlying their study is nevertheless valid, i.e. a physically meaningful steady state solution must be approachable at sufficiently large times from an arbitrary initial state. In this way also we eliminate the need for specifying a radiation condition for the steady state. We shall now exploit it further to determine the properties of such solutions in simple cases so as to throw additional light on the structure of the steady flow when $H > 0$.

Linearization for thin obstacles

We now assume that the obstacle is thin or the ground topography is shallow, i.e. $\epsilon \ll 1$, which is not expected to severely limit the validity of our conclusions regarding the scalings and certain general features of the flow (the far-field structure at $r \gg 1$ in particular).

With the substitutions $w = \epsilon w'$, $p = -y + \epsilon p'$, the governing equations (2.7) and (2.8) simplify for fixed H, after neglecting terms of $O(\epsilon^2)$, to

$$\frac{\partial p'}{\partial z} = -H\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial x}\right)w', \quad \frac{\partial w'}{\partial z} = H\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial x}\right)\nabla^2 p' \tag{2.11a}$$

together with the boundary conditions

$$w' = \partial f/\partial x \quad \text{at} \quad z = 0; \quad w' = 0 \quad \text{at} \quad z = 1; \tag{2.11b}$$

$$(p'_x, p'_y, w) \to (0, 0, 0) \quad \text{as} \quad r \to \infty, \quad t < \infty, \tag{2.11c}$$

with some initial conditions at $t = 0$. The transverse velocity u and v can be computed from p' (with remainders of order ϵRo) as

$$u = 1 - \epsilon\,\partial p'/\partial y, \quad v = \epsilon\,\partial p'/\partial x. \tag{2.12}$$

The two differential equations (2.11a) can be combined to yield

$$\left[H^2\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial x}\right)^2 \nabla_1^2 + \frac{\partial^2}{\partial z^2}\right] p' = 0, \tag{2.13}$$

which, except for the difference between ∇_1^2 and $\nabla^2 = \nabla_1^2 + \partial^2/\partial z^2$ is essentially the inertial-wave equation in the body-fixed co-ordinates. The solution of (2.13) has been studied by Lighthill (1967) and Cheng (1977), with emphasis on the limit $H \to \infty$ (after replacing z by $\hat{z} = z/H$). In this limit it was found that the disturbance of the flow due to an obstacle with finite displacement volume is, at large distances, largely confined within a wedge $x > 2^{\frac{3}{2}}|y|$ in which lee waves are set up whose wave length is roughly proportional to $\hat{z}/x$, with an amplitude of p' comparable to $l/\hat{z}$, undiminishing in the downstream direction ($x/y \to \infty$), and an amplitude of v diverging with x like x/z^2. Cheng (1977) is also able to describe how these lee waves become evanescent when $x < 2^{\frac{3}{2}}|y|$, and their behaviour near the caustic $x = 2^{\frac{3}{2}}|y|$, as well as the structure of the caustic itself. Here we shall extend the theory to cover $H = O(1)$ so as to bridge Cheng's (1977) solution with that for $H \ll 1$ and determine the solution properties both for two-dimensional and three-dimensional flows. We shall show that several properties found for $H \to \infty$ are in fact universal for all $H \neq 0$, especially the lee wave structure extending to a considerable distance downstream. On the other hand, a distinctly different, column-like feature emerges as a part of the solution for finite H. In view of the coexistence of this and the other (lee wave) features, the flow may be said to possess a *bimodal* structure.

Solutions to (2.11), or to (2.13) with $w' = \partial f/\partial x$ replaced by $\partial p/\partial z = -H\,\partial^2 f/\partial x^2$ in (2.11b), are studied for ridge-like obstacles in § 3, and for finite three-dimensional shapes in § 4. For convenience, the primes in p' and w' are omitted hereafter. We observe in passing that the linearization leading to (2.11) requires $\epsilon \ll 1$ under a fixed H.

3. The ridge

Solution for arbitrary profiles

When the obstacle takes the form of a narrow ridge, the flow in a plane normal to the ridge line becomes essentially two-dimensional, and if we take the y axis along the ridge, (2.10) reduces to

$$\frac{\partial p}{\partial z} = -H\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial x}\right)w, \quad \frac{\partial w}{\partial z} = H\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial x}\right)\frac{\partial^2 p}{\partial x^2}. \tag{3.1}$$

We notice at once that the boundary conditions on w alone do not fix p uniquely in this case, for there are solutions of (3.1) of the form

$$w = 0, \quad p = xf_1(t)+f_2(t)+f_3(x-t), \tag{3.2}$$

where f_1, f_2, f_3 are arbitrary functions. Of these f_3 represents a wave travelling with the fluid, and therefore is carried far away from the obstacle at large values of t. The other two terms represent trapped disturbances, and it is conceivable that they make contributions to p, and even to v, in the neighbourhood of the obstacle without affecting w. However, these functions cannot be arbitrarily added to the solution desired, since they contradict the requirement (2.11c) for all $t \neq \infty$. We shall return to this point below.

Let $F(\omega)$ be the Fourier transform of the displacement $f(x)$ with respect to x [cf. (2.11 b)], and let $\bar{p}(x,z;s)$ denote the Laplace transform of p with respect to t. Further, let us take the initial conditions of the problem, at $t = 0$, to be $w = 0$, $\partial^2 p/\partial x^2 = 0$ for all x, z. It is not expected that this choice of initial condition is significant for the final solution as $t \to \infty$. The solution of (3.1) satisfying the boundary condition (2.11 b) is

$$\bar{p} = \frac{1}{2\pi s}\int_{-\infty}^{\infty} e^{i\omega x}F(\omega)\frac{\cos\{\omega H(\omega-is)(1-z)\}}{\sin\{\omega H(\omega-is)\}}d\omega. \tag{3.3}$$

The contour of integration is the real axis of ω except for an indentation above or below the pole at $\omega = 0$. The uncertainty here is associated with the apparent non-uniqueness of p discussed in connexion with (3.2). To fix matters we shall, to begin with, choose the contour to pass below the origin; later we shall give the solution when the contour passes above $\omega = 0$, but below the pole at $\omega = is$, of course, the velocity field is shown to be unaffected by the choice of contour.

Example: top-hat ridge

As a first example, let us consider a top-hat ridge in which $f(x) = 1$ if $|x| < 1$, and is zero otherwise. Singular behaviour is to be expected in the solution in the neighbourhood of $x^2 = 1$ but is merely a local phenomenon and, as we shall see, may easily be eliminated by choosing a smoother form for f. As a matter of fact, the singularity is rather weak in spite of the discontinuities in the surface elevation at $x = \pm 1$. For this case,

$$F(\omega) = \frac{1}{i\omega}(e^{i\omega}-e^{-i\omega}) \tag{3.4}$$

and, in order to evaluate (3.3), we consider

$$\bar{p}_+(x,z;s) = \frac{1}{2s\pi i}\int_{-\infty}^{\infty}\frac{\exp(i\omega(x+1))}{\omega}\frac{\cos[\omega H(\omega-is)(1-z)]}{\sin\omega H(\omega-is)}d\omega. \tag{3.5}$$

This integral is evaluated by completing the contour with a semi-circle, above the real axis if $x+1 > 0$ and below the real axis if $x+1 < 0$. Further, since our interest is entirely in the solution properties when $t \gg 1$, we take s real and positive with $sH^{\frac{1}{2}} \ll 1$. The value of $\bar{p}_+$ then depends on the residues of the integrand at its poles, i.e.

$$-\frac{1}{2\pi i H}\left[\frac{x+1}{s^2}-\frac{1}{s^3}\right] \quad \text{at} \quad \omega = 0, \qquad -\frac{\exp[-s(x+1)]}{2i\pi s^3 H} \quad \text{at} \quad \omega = is,$$

$$-\frac{\exp(\mp\omega_n(1+x))\cos n\pi z}{4\pi^2 nsi} \quad \text{at} \quad \omega = \pm i\omega_n + O(s)$$

and

$$\frac{\exp(\pm i\omega_n(1+x))\cos n\pi x}{4\pi^2 nsi} \quad \text{at} \quad \omega = \pm\omega_n + \frac{is}{2} + O(s^2), \tag{3.6}$$

where $\omega_n = (n\pi/H)^{\frac{1}{2}}$ and n is a positive integer. The residues at those poles for which $\operatorname{Im}\omega > 0$ are relevant to $\bar{p}$ when $x+1 > 0$, and the rest when $x+1 < 0$. Hence,

$$\left.\begin{aligned} \bar{p}_+ &= -\frac{1}{H}\left(\frac{1}{s^3}-\frac{x+1}{s^2}\right)-\frac{1}{Hs^3}\exp[-s(x+1)] \\ &\quad -\sum_{n=1}^{\infty}\frac{\exp[-\omega_n(1+x)]}{2\pi ns}\cos n\pi z+\sum_{n=1}^{\infty}\frac{\cos\omega_n(1+x)\cos n\pi z}{\pi ns}, \quad \text{if} \quad x+1 < 0; \\ \bar{p}_+ &= \sum_{n=1}^{\infty}\frac{\exp[\omega_n(1+x)]\cos n\pi z}{2\pi ns}, \quad \text{if} \quad x+1 < 0. \end{aligned}\right\} \tag{3.7}$$

Similar results are obtained when $x+1$ is replaced by $x-1$ in (3.7), and so, on subtracting the two and examining the residue at $s = 0$, we obtain the ultimate pressure distribution for $t \to \infty$ due to the obstacle defined by (3.4). This is

$$p \sim \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\exp(\omega_n x)\sinh\omega_n\cos n\pi z \quad \text{if} \quad x < -1, \tag{3.8a}$$

$$\begin{aligned} p &\sim -\frac{(x+1)^2}{2H}+\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\exp(-\omega_n)\cosh(\omega_n x)\cos n\pi z \\ &\quad -\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{\cos\omega_n(1+x)\cos n\pi z}{n} \quad \text{if} \quad |x| < 1, \end{aligned} \tag{3.8b}$$

and

$$\begin{aligned} p &\sim -\frac{2x}{H}+\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\exp(-\omega_n x)\sinh\omega_n\cos n\pi z \\ &\quad -\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\omega_n\sin(\omega_n x)\cos n\pi z \quad \text{if} \quad x > 1. \end{aligned} \tag{3.8c}$$

All the above series converge.

This solution, with the choice of the contour passing below $\omega = 0$, satisfies the upstream requirement (2.11c). The consequence of taking the alternative contour is to transfer the residue of the pole at $\omega = 0$ to the lower half-plane for $x+1 > 0$, and the final effect is to add a term t/H to the final solution (3.8); thus there is no difference to the velocity field.

When $x < -1$, the series for p converges and the sum decays exponentially as $x \to -\infty$; the upstream influence of the ridge can be seen from the argument

$$\omega_n x = (n\pi/H)^{\frac{1}{2}}x$$

in (3.8a) to be limited to a distance of $x = O(H^{\frac{1}{2}})$. As $H \to \infty$, this distance formally increases indefinitely but the magnitude of p upstream becomes vanishingly small by virtue of the factor $\sin h\omega_n \sim \omega_n = O(H^{-\frac{1}{2}})$ in (3.8a). On the plane $z = 0$, the solution develops a logarithmic singularity on approaching the leading edge

$$p \sim \frac{1}{2\pi}\left(\log\frac{|x+1|^2\pi}{H}+\gamma\right), \quad \text{as} \quad x \to -1, \tag{3.9}$$

where γ is Euler's constant, but remains bounded for $z > 0$. This is not surprising, since the theory cannot be expected to make physical sense at the corners of the top-hat and the singularity disappears if the corners are smoothed off.

Downstream, i.e. $x > 1$, the pressure field (3.8c) consists of three parts. First, there is a linear term in x with a corresponding uniform velocity $-2/H$ in the y direction. In physical terms, this velocity is $-4\Omega l^2\epsilon/h$, i.e. $-Af/h$, where A is the cross-sectional area of the ridge and $f = 2\Omega$ is the Coriolis parameter. Assuming that inertial terms may be neglected, one can make a rather simple argument leading to this result (e.g. Prandtl 1952; Batchelor 1967). The second contribution arises from terms which decrease exponentially as x is increased, and play a similar role to those for $x < -1$. The third contribution consists of an infinite series of oscillatory terms which do not decay as $x \to \infty$, and its analytical behaviour will be studied further in the appendix. Thus, the notion that the effect of the ridge on a low Rossby number inviscid flow is to deflect its direction by means of the velocity component $-Af/h$ parallel to the ridge is seen to be an oversimplification. As $H \to 0$ however this effect dominates, since the lee wave contribution to the pressure is at most $O[\log(x/H)]$ and to the velocity field is $O(H^{-\frac{1}{2}})$. An immediate consequence is that when $H \ll 1$ the linearization (2.11) is strictly only valid if $\epsilon/H \ll 1$; in view of (2.9) however it is also valid if $\epsilon = O(H)$ and closed streamlines have not appeared in the flow field.

Example: smooth ridges

The integral in (3.3) can also be evaluated on a series for smooth ridges and is then free of edge singularities such as (3.9). For example if $f(x) = (1-x^2)$ for $|x| < 1$ and $f(x) = 0$ for $|x| > 1$ so that

$$F(\omega) = \frac{2}{\omega^2}\left(\frac{\sin\omega}{\omega} - \cos\omega\right). \tag{3.10}$$

The only significant charge in the solution is that the coefficient n^{-1} in the series of (3.8) is replaced by another $O(n^{-2})$ as $n \to \infty$, thus improving convergence properties and ensuring that p is continuous at $x = \pm 1$. An important consequence is that the lee waves are weaker as $H \to 0$, their contribution to the velocity being $O(1)$. In fact the smoother the ridge the weaker the lee waves are when $H \ll 1$; thus if

$$f(x) = \frac{1}{1+x^2}, \quad F(\omega) = \pi\exp[-|\omega|] \tag{3.11}$$

the pressure field is smooth everywhere and the lee waves are exponentially small as $H \to 0$.

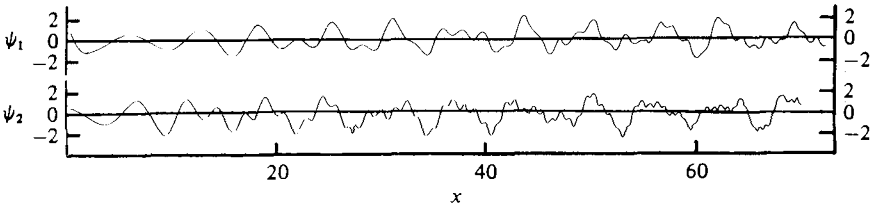

FIGURE 1. The function $\psi_1(x)$ for the lee wave behind a top-hat ridge at the bottom ($z = 0$) and the function $\psi_2(x)$ for the lee wave behind a top-hat ridge at the top ($z = 1$).

The lee waves

The lee wave contribution to the pressure on the plane $z = 0$, downstream from a top-hat ridge, is given by the last series of (3.8 c). On the plane $z = 0$, it can be written as

$$\frac{1}{\pi}\psi_1\left[(x+1)\left(\frac{\pi}{H}\right)^{\frac{1}{2}}\right]-\frac{1}{\pi}\psi_1\left[(x-1)\left(\frac{\pi}{H}\right)^{\frac{1}{2}}\right], \tag{3.12a}$$

where

$$\psi_1(x) = \sum_{n=1}^{\infty}\frac{1}{n}\cos(xn^{\frac{1}{2}}). \tag{3.12b}$$

This function has not received much attention in the past, but Hardy & Littlewood (1913) have established its convergence for $x > 0$. At large values of n, the terms of the series may be replaced by integrals, and so the sum can be explicitly found, partly by the direct addition of a large number (1200) of the terms and partly by using an asymptotic formula for the remainder. The graph of $\psi_1(x)$ is displayed in figure 1 (upper curve); it appears to be smooth and oscillatory, with no particular frequency, and does not die out as $x \to \infty$. On the other hand, it is shown in the appendix that $\psi_1(x) < 2\log x + 10$ for all $x > 1$. It may be differentiated once, even though term-by-term differentiation of the series fails to converge and it is even possible that ψ_1 is quite smooth. On the other hand, we know that when $n^{\frac{1}{2}}$ is replaced by n in (3.12b), the resulting function is singular at $x = 2n\pi$ ($n = 0, 1, 2, \ldots$).

The lee wave structure for $z \neq 0$ may be exemplified by taking $z = 1$. The contribution to the pressure is similar to (3.12) with ψ_1 replaced by ψ_2, where

$$\psi_2(x) = \sum_{n=1}^{\infty}\frac{(-1)^n}{n}\cos(xn^{\frac{1}{2}}). \tag{3.13}$$

The series again converges, and a graph of the function is shown also in figure 1 (lower curve). Again it appears to be smooth and irregularly oscillatory and does not die out as $x \to \infty$.

The lee wave structure is seen from above to fill the entire region of flow downstream of the ridge, and (unlike the implication of HIL) there is no tendency for the disturbances to concentrate in a finite region above and behind the ridge. The velocity field associated with the lee waves behind its ridge is accordingly $O(H^{-\frac{1}{2}})$, whereas the lateral drift velocity is of $-z/H$. Thus, lee waves prevail when H is large, and the lateral drift dominates when H is small.

4. Three-dimensional obstacles

We now consider thin obstacles which are three-dimensional, i.e. f is a function of x and y in (2.11*b*). We define

$$F(\omega, \sigma) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp(-i\omega x - i\sigma y) f(x,y)\,dx\,dy, \tag{4.1}$$

and then the formal solution for the Laplace transform of p to (2.11) is

$$\bar{p} = \frac{1}{4\pi^2 s}\iint_{-\infty}^{\infty} \exp(i\omega x + i\sigma y)\frac{\omega F(\omega,\sigma)}{(\omega^2+\sigma^2)^{\frac{1}{2}}}\left\{\frac{\cos[H(\omega - is)(\omega^2+\sigma^2)^{\frac{1}{2}}(1-z)]}{\sin[H(\omega-is)(\omega^2+\sigma^2)^{\frac{1}{2}}]} - \frac{1}{H(\omega-is)(\omega^2+\sigma^2)^{\frac{1}{2}}}\right\} d\omega\,d\sigma + \bar{p}_0(x,y), \tag{4.2}$$

where $\bar{p}_0$ is

$$\bar{p}_0 = \frac{1}{4\pi^2 s}\iint_{-\infty}^{\infty} \exp(i\omega x + i\sigma y)\frac{\omega F(\omega,\sigma)}{H(\omega-is)(\omega^2+\sigma^2)}\,d\omega\,d\sigma. \tag{4.3}$$

As $t \to \infty$ the integral in (4.3) leads easily to

$$\nabla_1^2 p_0 = -f(x,y)/H \tag{4.4}$$

so that this contribution to the pressure may be identified with Ingersoll's (1969) geostrophic form when $H \ll 1$. The solution of (4.4) for the spherical cap

$$f(x,y) = 1 - r^2 \quad \text{if} \quad r < 1, \quad f(x,y) = 0 \quad \text{if} \quad r > 1, \tag{4.5}$$

$$F(\omega,\sigma) = \frac{4\pi}{R^3}[2J_0'(R) + RJ_0(R)], \tag{4.6}$$

where $r^2 = x^2+y^2$, $R^2 = \omega^2+\sigma^2$ and J_0 is a Bessel function, is

$$\left.\begin{aligned} p_0 &= -\tfrac{1}{4}H^{-1}\log r + A_0, \quad r > 1,\\ p_0 &= \frac{1}{H}\left(\tfrac{1}{16}r^4 - \tfrac{1}{4}r^2 + \tfrac{3}{16}\right) + A_0, \quad r < 1,\end{aligned}\right\} \tag{4.7}$$

where A_0 is independent of position. The corresponding motion in the x, y planes is a generalized Rankine vortex with constant circulation outside the cap and variable vorticity above it. The normal component w is linear in z above the cap and zero outside it, again in line with Ingersoll's form (2.9) as $H \to 0$.

The solution of (4.4) can also be found for smooth bodies, a simple example being the topographical feature

$$f = \frac{1}{(1+x^2+y^2)^2}, \quad F = \pi R K_1(R) \tag{4.8}$$

and

$$p_0 = -\tfrac{1}{4}H^{-1}\log(1+x^2+y^2) + \text{const.} \tag{4.9}$$

Now let us examine the contribution from the double integral of (4.2), designated hereafter as p_1. For obstacles which decay monotonically to zero as x^2, $y^2 \to \infty$, the poles of F may be expected to lie off the real axis of ω, when σ is real, and their effect on the flow field confined to their neighbourhood. A similar remark applies to the zeroes of $\sin[H(\omega - is)(\omega^2+\sigma^2)^{\frac{1}{2}}]$ qua function of ω when s, σ are real, except those in the neighbourhood of the real axis as $s \to 0$. Hence, we may take the solution in § 3 to be typical of the solution here upstream of the obstacle, i.e. the disturbances decay exponentially. Above the obstacle, there are no features of this part of the solution of special interest. Downstream, in addition to the exponential decay from the poles well away from the real axis of ω, there is a lee wave structure determined by the poles lying just above the real axis of ω when s is small. These make a contribution $p_1(x,y)$ to p in the limit $t \to \infty$, where

$$p_1(x,y) = \frac{i}{2\pi H}\sum_{n=1}^{\infty}\cos n\pi z\int_{-\infty}^{\infty}\frac{\lambda^3}{1+\lambda^4}F(\omega_n\lambda,\omega_n\mu)\exp(i\omega_n\lambda x+i\omega_n\mu y)\,d\mu$$
$$-\frac{i}{2\pi H}\sum_{n=1}^{\infty}\cos n\pi z\int_{-\infty}^{\infty}\frac{\lambda^3}{1+\lambda^4}F(-\omega_n\lambda,\omega_n\mu)\exp(-i\omega_n\lambda x+i\omega_n\mu y)\,d\mu, \tag{4.10}$$

and

$$\omega_n = (n\pi/H)^{\frac{1}{2}}, \quad \lambda = [-\tfrac{1}{2}\mu^2+(1+\tfrac{1}{4}\mu^4)^{\frac{1}{2}}]^{\frac{1}{2}}. \tag{4.11}$$

It is immaterial whether we regard λ as positive for all μ or permit it to change sign; it is only necessary to rearrange the integrals in (4.10). Since there are advantages to be gained in simplicity from taking λ to be analytic, we now change the definitions slightly and write

$$\lambda\mu = (1-\lambda^4)^{\frac{1}{2}}, \quad -1<\lambda<1. \tag{4.12}$$

Then a typical integral of (4.10) to be evaluated is

$$Q_n(x,y) = \int_{-\infty}^{\infty}\frac{\lambda^3}{1+\lambda^4}F(\omega_n\lambda,\omega_n\mu)\exp[i\omega_n(\lambda x+\mu y)]\,d\mu \tag{4.13}$$

and we are especially interested in its value when $\omega_n x \gg 1$. The dominant contributions to (4.13) arise from the neighbourhood of the cols at

$$x = \frac{y}{\lambda^2}\frac{1+\lambda^4}{(1-\lambda^4)^{\frac{1}{2}}}. \tag{4.14}$$

Suppose first that $y>0$ and $x^2>8y^2$, i.e. inside the Lighthill (1967) wedge.† Then the relevant saddles are at $\lambda = \pm\lambda_1, \pm\lambda_2$, where λ_1, λ_2 are real and

$$\lambda_{1,2}^4 = \frac{1}{2(x^2+y^2)}[x^2-2y^2\pm x(x^2-8y^2)^{\frac{1}{2}}]. \tag{4.15}$$

The integral in (4.13) may now be evaluated using the method of steepest descents provided that F is bounded (e.g. ridges are excluded). We find that p_1 is dominated by the sum of two terms when $\omega^{\frac{1}{2}}x \gg 1$, one of which is

$$\frac{i\lambda_1^{\frac{3}{2}}(1+\lambda_1^4)^{\frac{1}{4}}}{2H(3\lambda_1^4-1)^{\frac{1}{2}}(\pi x)^{\frac{1}{2}}}\sum_{n=1}^{\infty}\frac{\cos x\pi z}{\omega_n^{\frac{1}{2}}}\{\exp(iX_{n1})F(\omega_n\lambda_1,\omega_n\mu_1)$$
$$-\exp(-iX_{n1})F(-\omega_n\lambda_1,-\omega_n\mu_1)\}, \tag{4.16}$$

where $X_{n1} = \omega_n(\lambda_1 x+\mu_1 y)-\frac{1}{4}\pi$. The other is the same except that $(3\lambda_1^4-1)$ is replaced by $1-3\lambda_2^4$, λ_1 and μ_1 by λ_2 and μ_2 and $X_{n2} = \omega_n(\lambda_2 x+\mu_2 y)+\frac{1}{4}\pi$. In computing (4.16) it should be remembered that only saddles on the real axis of ω are relevant so that the contribution from an integral like (4.13), with λ changed in sign, is negligible. A formula similar to (4.13) may be obtained when $y<0$ and $x^2>8y^2$.

† The wedge-shaped front $y = \pm x/2^{\frac{3}{2}}$ is identical to that in Kelvin's ship waves.

On the other hand, when $x^2 < 8y^2$, the possible cols are eight in number and all are complex, being given by

$$\lambda^4 = \frac{1}{2(x^2+y^2)}[x^2-2y^2 \pm ix(8y^2-x^2)^{\frac{1}{2}}]. \tag{4.17}$$

Half of them may be excluded depending on the sign of y, and of the remainder it is possible to choose the path of integration in (4.13) to pass only through those two which ensure that the integral is exponentially small. The evaluation of the integral is a technical task and would follow closely the earlier study of Cheng (1977), which also provides a discussion of the special features of the integrals like (4.13) when $x^2 \approx 8y^2$.

The chief features of the steady-state flow past an obstacle, whether of compact support or not, are first a cylindrical flow in the x, y plane with a vorticity proportional to the elevation $f(x,y)$ and inversely proportional to H, which has no tendency to tilt and an axial velocity component varying linearly from a prescribed form at $z = 0$ to zero at $z = 1$. Secondly, there is a distinct upstream influence extending a distance $= O(H^{\frac{1}{2}})$, decaying exponentially with x. There is a similar contribution downstream, but this is swamped by the third feature: far downstream of the obstacle (if it is finite in extent, and broadly downstream of the highest point if infinite), there is a complicated lee wave structure bounded by the wedge surface $x = \pm 2(2y)^{\frac{1}{2}}$. Outside this wedge, the lee waves make an exponentially small contribution to the flow field, but inside they take on a quasi-random oscillatory character in the variable $x/H^{\frac{1}{2}}$ with an amplitude decaying like $x^{-\frac{1}{2}}$ as x increases, for all finite H.

The first, cylindrical part of the solution, i.e. p_0, cannot be cancelled completely by p_1 which possesses the second and the third features, because at large transverse distance the p field is dominated by the cylindrical p_0 which behaves like the stream function of a vortex and does not decay to zero at infinity. The lee wave velocity field is however dominant in the Lighthill wedge when $x \gg 1$. In this sense, the solution may be said to possess a *bimodal* structure.

When $H \gg 1$, $\omega_n \ll 1$ for finite n, and in the calculation of the series (4.16) for the lee waves F is virtually a constant equal to the volume of the obstacle. The series is however now even less convergent than (3.12) and may even diverge. In this case it would be necessary to examine more carefully the terms of the series when $n = O(H)$. The lee wave pressure is certainly $\geqslant O(H^{-\frac{3}{4}})$ and the velocity field $\geqslant O(H^{-\frac{1}{4}})$ so that they dominate the geostrophic motion which is inversally proportional to H.

When $H \ll 1$ the amplitude of the lee waves is linked as in two dimensions to the degree of smoothness of the obstacle. If they are perfectly smooth [e.g. (4.8)] the amplitude is exponentially small while for a spherical cap with a corner at $r = 1$ [(4.5), (4.6)] $p_1 = O(1)$ and even so is much weaker than the geostrophic contribution, which is $O(H^{-1})$.

5. Discussion

In this paper we have considered relative fluid motion about a shallow topographic feature when the ratio of its lateral dimension to the depth of the rotating fluid is comparable with the Rossby number, i.e. $H = O(1)$. For an inviscid fluid, the unsteady flow field asymptotes to a bimodal structure as $t \to \infty$, one part consisting of an untilted cylindrical motion similar to a Taylor column; the other giving a smoothly decaying motion upstream, together with a lee wave system, originating at and above the feature and filling a wedge region far downstream. In no case does the solution require a topography with compact support. Similar remarks apply to two-dimensional component flows above narrow ridges. Our analyses have been limited to inviscid problems linearized for thin obstacles, although the nonlinear basic equations derived, (2.6)–(2.9), are equally applicable to thick bodies.

Damping mechanism and Ekman pumping

Even for thin obstacles, the viscous damping of the lee waves and suction/pumping effect of the Ekman boundary layer may modify the inviscid solution significantly. Consider the ridge analysis as an example. First,e xtra terms have to be added to the right-hand sides of (3.1), namely

$$\frac{HE}{2Ro}\frac{\partial^2 w}{\partial x^2} \quad \text{and} \quad -\frac{HE}{2Ro}\frac{\partial^4 p}{\partial x^4}, \tag{5.1}$$

respectively, where $E = \nu/\Omega l^2$ is the Ekman number and ν is the kinematic viscosity. A typical lee wave now has the form $w = \sin(n\pi z)\exp(\alpha x)$, and if $Ro \gg E$

$$\alpha = i\omega_n - \frac{n\pi E}{4HRo} + O(E^2) \tag{5.2}$$

and the real part of α is the inverse of the e-folding distance. Second, Ekman layers are set up near each of the bounding planes and the pumping conditions may be written as

$$w = \pm \tfrac{1}{2}E^{\frac{1}{2}}\frac{\partial^2 p}{\partial x^2} \quad \text{at} \quad z - \tfrac{1}{2} = \pm \tfrac{1}{2}, \tag{5.3}$$

from which the e-folding rate of decay of the lee waves is $2HE^{-\frac{1}{2}}$. Thus for these lee waves to be seen in an experiment, it is essential that

$$\frac{E}{HRo} \ll 1, \quad \frac{E^{\frac{1}{2}}}{H} \ll 1, \tag{5.4}$$

This last condition has a bearing on the question of the existence of a column with stagnant trapped fluid discussed in § 2. If such a column developed when $\epsilon = O(1)$ the $E^{\frac{1}{4}}$ layer at its outer boundary must separate. The condition for this is $E^{\frac{1}{2}}/H < 2$ and follows from Boyer's (1970*a*, *b*) experiment and the theoretical studies of Walker & Stewartson (1972, 1974). Hence such a column can be expected to generate a wake.

Lee waves and the bimodal structure

Cheng (1977) has shown that lee waves with an undiminishing strength are salient features of the far field ($z \gg H$, $x \gg y$) in the problem for an infinite domain. The present study confirms the prevalence of lee waves and shows that this feature remains in a container of finite depth, so long as $H \neq 0$. On the other hand, another distinctly different feature, the cylindrical mode, emerges. At large transverse distances, the p' field is dominated at $H = O(1)$ by the untilted, unwavy cylindrical mode p_0, which behaves generally like the stream function of a vortex and is responsible for the stream deflection at the ridge in the two-dimensional case. We note that, unlike the solution for $H \to \infty$, the pressure perturbation in the lee wave mode for a finite H decays (roughly like $x^{-\frac{1}{2}}$) far downstream; on the other hand, the velocity field at large distances is dominated by lee waves, unless H becomes very small. The cylindrical mode p_0 is the same as Ingersoll's geostrophic flow (1969) valid when $\epsilon \leqslant O(H) \ll 1$ and under these conditions the lee waves are weak.

There is little evidence for the existence of lee waves at finite values of H in earlier experiments, the primary reason being the relatively large value of E in the operating conditions, but they are clearly present in the recent studies by Maxworthy (1977) on the effects of ground topography. Here a ridge, spanning radially across the annulus, is mounted on the base of the tank, which rotates at a slightly lower rate than the top and the annular channel walls. Distinct meandering streamline patterns extending many chord lengths downstream of the ridge are consistently found on the photo records for tests corresponding to the larger values of H. A quantitative comparison with our ridge theory (in § 3) is not possible at the present time, because the finite width of the annular channel and the breakdown of the solution at impermeable end walls has not been accounted for in the two-dimensional theory. Figure 2 (plate) shows typical streamline photographs (kindly provided by Dr T. Maxworthy) selected from unpublished records for high aspect ratio ridges not shown in Maxworthy's (1977) original paper. The cross-section of the lenticular obstacle has a thickness ratio $\epsilon \approx \frac{1}{4}$. In figure 2(*a*) $H = 0{\cdot}155$, $Ro = 0{\cdot}071$, $E = 0{\cdot}00019$ and lee waves are not very apparent. They are more in evidence in figure 2(*b*), for which $H = 0{\cdot}46$, $Ro = 0{\cdot}071$ and $E = 0{\cdot}00234$. An almost identical pattern is seen in figure 2(*c*), for which $H = 0{\cdot}45$, $Ro = 0{\cdot}069$ and $E = 0{\cdot}00060$. The pattern observed is evidently consistent with, and predictable by, our *inviscid* theory; this is not surprising, in spite of the presence of some viscous damping, since lee waves in the two-dimensional case are stronger than those in three dimensions (3-D), and, unlike their 3-D counterparts, do not diminish downstream. We note that figure 2 confirms not only the persistence of lee waves behind a ridge, but also the deflexion of the mean fluid trajectories corresponding to p_0. Similar photographic records have also been made by Maxworthy in an unpublished related study for finite three-dimensional obstacles.

Questions on tilting

In HIL (1968), a tilting of Taylor column above a sphere with a small (mean) angle $\frac{3}{2}Ro$ is deduced from experimental data, and an explanation is given on the basis of Lighthill's (1967) study on inertial waves in the far field. We shall discuss the questions on tilting in the far field and in HIL's experiment as two separate issues in the light of the foregoing analysis.

From Cheng's (1977) work, one may quite readily see that lines or curves along which *certain* flow properties are invariant (if they are properly defined) have a backward tilting angle comparable to Ro (on account of the bias toward downstream), but there is no 'tilting of the Taylor column' in the sense of HIL. This is because: (i) the lee wave trains with *undiminishing* strength continue far beyond the region covered by rays $x/Hz = O(1)$ (like a curtain); thus, in its outward extension, the field above the Taylor column loses its solitary feature, and therefore a column, tilted or not, does not exist in the far field; (ii) the tilting angles differ among different flow properties and are not uniform along wave crests; therefore, a distinct tilting angle does not exist. Whereas the equations used in our analyses are basically similar to those in Lighthill's (1967) study, the different conclusion of HIL (1968) regarding tilting results mainly from the assumption implicit in their argument that disturbances do not spread far beyond $x/Hz = O(1)$. It is interesting to observe that, for the case of a finite H, the part of p (not $v \approx \partial p/\partial x$) contributed by the lee wave system has a decaying amplitude at large x like $x^{-\frac{1}{2}}$, and one could very well interpret it to be a 'tilted column' if not for the presence of the other untilted, cylindrical mode, p_0, which dominates the far field of p.

We return now to the tilting experimentally observed in HIL. The values of the parameter H (estimated from table 3 in HIL) ranges from 0·34 to 5·74, and the flow should not be regarded as approaching the case $H \to \infty$ analysed by Lighthill (1967) and Cheng (1977), especially because the tilting data are not taken near the top of the tank. For a blunt obstacle, the study in this case would call for use of the nonlinear equations (cf. § 2), thus the remarkable agreement of tilting angles estimated by Lighthill's (1967) linear theory (appendix HIL) and that recorded in the experiment therefore can not be regarded as evidence supporting the theory. It is interesting to note that lee waves are not apparent in the experiments of both HIL (1968) and Hide & Ibbetson (1966), but the requirements for an inviscid model are not satisfied there. In HIL, $0{\cdot}05 < H < 5$, $0{\cdot}002 < Ro < 0{\cdot}02$, $0{\cdot}003 < E < 0{\cdot}03$, and taking median values, $E/HRo \sim 2E^{\frac{1}{2}}/H \sim 0{\cdot}2$. In Hide & Ibbetson, $H \sim 0{\cdot}03$, $E \sim 0{\cdot}0005$, $Ro \simeq 0{\cdot}008$ so that $E/HRo \sim 2$, $E^{\frac{1}{2}}/H \sim 1{\cdot}5$. In order to make a valid comparison with these experiments it is desirable to obtain a numerical solution of the nonlinear equations (2.7)–(2.9) incorporating the effect of viscosity by adding terms similar to (5.1), and by modifying the boundary conditions on w with terms similar to (5.3), which reflect the dependence of p on y as well as x. A useful start has been made by James (1977, private communication) by neglecting (2.7) and, instead, assuming that w is linear in z. The flow patterns obtained show an encouraging agreement with some of the observations.

The authors are grateful to Dr S. N. Brown for summing the series in § 3 and providing the figures, and Mr H. Kestelman and Professor G. L. Watson for providing the arguments which led to the proofs in the Appendix. The plates in figure 2 were obtained from unpublished photos furnished by Professor T. Maxworthy, who called our attention to the presence of lee waves in his experiments.

Appendix

We wish to prove that

$$f(x) = \sum_{n=1}^{\infty} \frac{\exp(ixn^{\frac{1}{2}})}{n}, \quad x > 0, \tag{A 1}$$

is a continuous function of x and to obtain a bound on its magnitude. Define

$$J(x,n) = \int_0^n \exp(ix\xi^{\frac{1}{2}})\, d\xi = \frac{2n^{\frac{1}{2}}}{ix}\exp(ixn^{\frac{1}{2}}) + \frac{2}{x^2}(\exp(ixn^{\frac{1}{2}}) - 1) \tag{A 2}$$

and

$$S(x,n) = \sum_{m=1}^{n} \exp(ixn^{\frac{1}{2}}), \tag{A 3}$$

so that

$$f(x) = \sum_{n=1}^{\infty} \frac{S(x,n)}{n(n+1)}. \tag{A 4}$$

Now, if m is the least integer greater than or equal to ξ,

$$|\exp(ixm^{\frac{1}{2}}) - \exp(ix\xi^{\frac{1}{2}})| \leqslant \frac{x}{2\xi^{\frac{1}{2}}} \tag{A 5}$$

and so

$$|S(x,n) - J(x,n)| \leqslant \frac{x}{2}\int_0^n \xi^{-\frac{1}{2}} d\xi = xn^{\frac{1}{2}}. \tag{A 6}$$

From (A 3) it follows that

$$|S(x,n)| \leqslant n \tag{A 7}$$

and from (A 2), (A 6) that

$$|S(x,n)| \leqslant xn^{\frac{1}{2}} + \frac{2n^{\frac{1}{2}}}{x} + \frac{4}{x^2}. \tag{A 8}$$

Hence $S(x,n) = O(n^{\frac{1}{2}})$ for all fixed $x > 0$ and so $f(x)$ is uniformly convergent in any positive interval of x. Consequently $f(x)$ is continuous in the open interval $(0, \infty)$.

Further, if N is the least integer exceeding x^2 and $x \geqslant 1$ we may estimate $|f(x)|$ by using (A 7) for $n \leqslant N$ and (A 8) for $n > N$. Then

$$\begin{aligned} |f(x)| &\leqslant \sum_{n=1}^{N} \frac{n}{n(n+1)} + \left(x + \frac{2}{x}\right) \sum_{N+1}^{\infty} \frac{1}{n^{\frac{1}{2}}(n+1)} + \frac{4}{x^2} \sum_{N+1}^{\infty} \frac{1}{n(n+1)} \\ &\leqslant \log N + \gamma - 1 + \frac{2(x^2+2)}{xN^{\frac{1}{2}}} + \frac{4}{Nx^2} \\ &\leqslant 2\log x + 10. \end{aligned} \tag{A 9}$$

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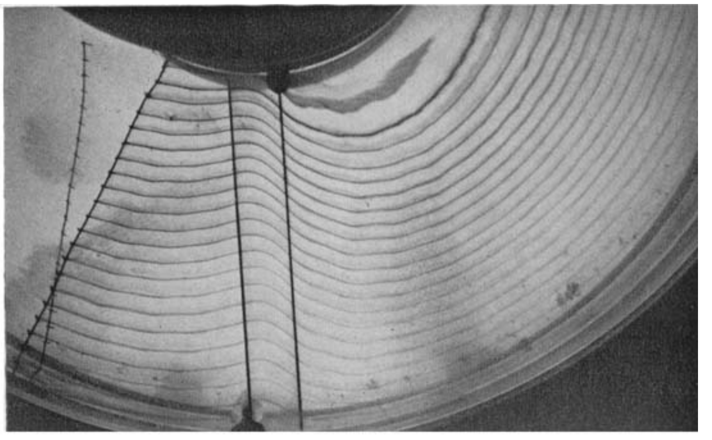
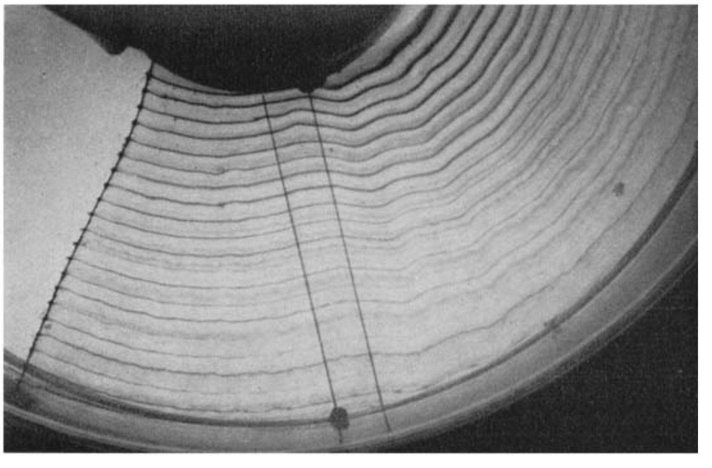
(*a*)



(*b*)



(*c*)

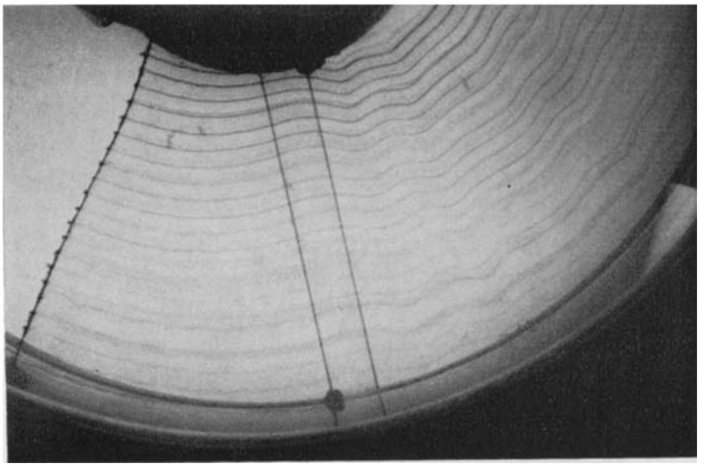


FIGURE 2. Photographs of streamline patterns over a lenticular transverse ridge at the base of a rotating tank. The flow over the ridge model is driven by a differentially rotating top plate. The height/width ratio of the obstacle is 0·261. (*a*) $Ro = 0{\cdot}071$, $H = 0{\cdot}155$, $E = 0{\cdot}00019$; (*b*) $Ro = 0{\cdot}071$, $H = 0{\cdot}46$, $E = 0{\cdot}00234$; (*c*) $Ro = 0{\cdot}069$, $H = 0{\cdot}451$, $E = 0{\cdot}00060$ (courtesy of T. Maxworthy). Note that the Ekman number E is based on the length l.